Homework # 4

Exercise 1: Uncertainty principle in everyday life

Suppose we measure the velocity of an object with a precision of $\Delta v = 1\,\text{mm/s}$. According to the Heisenberg uncertainty principle for position and momentum,

$$\Delta x \Delta p \geq \frac{\hbar}{2}\,,$$

what is the uncertainty on the position of:

1. Sinner's tennis ball, which weights around 60 grams?
2. an electron?

Hint: $h = 6.63 \times 10^{-34} J \cdot s$ *and* $m_e = 9.109 \times 10^{-31} kg$.

Exercise 2: Variational principle - Electron in a box

Consider an electron confined in a one-dimensional box of length a $(0 \leq x \leq a)$ with infinitely high walls.

1. Consider the variational principle of quantum mechanics and determine a minimum of the energy for the trial wave function:

   $$\Psi(x) = \alpha \sin\left(\frac{2\pi x}{a}\right) + \beta \sin\left(\frac{4\pi x}{a}\right),$$

   where α and β are real variational parameters. Consider the trial wave function to be normalized to unity and to satisfy the correct boundary conditions.

   Hint: $2 \sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$.

2. Compare the obtained value of the energy with the true ground-state energy $E_{n=1} = \frac{\hbar^2 \pi^2}{2ma^2}$. Can the system reach its ground state using the trial wave function given above? Why?

   Hint: The Hamiltonian in this system is given by $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$.

Exercise 3: Angular momentum operator

Consider the angular momentum operator in three dimensions $\hat{L} = \hat{r} \times \hat{p}$. Verify that the components $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ satisfy:

$$[\hat{r}_i, \hat{L}_j] = i\hbar\,\varepsilon_{ijk}\hat{r}_k,$$

$$[\hat{p}_i, \hat{L}_j] = i\hbar\,\varepsilon_{ijk}\hat{p}_k,$$

where each of indices i, j, k runs over three indices x, y, z (note that $r_x = x$, $r_y = y$, $r_z = z$) and ε_{ijk} is the Levi-Civita symbol, that reverses the sign under a pairwise interchange of the indices:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3) \\ 0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases} .$$

Here $1, 2, 3$ refers to the components x, y, z .

Hint: Use the rules that you have learnt during classes

$$[\hat{r}_i, \hat{r}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0 \qquad \forall i, j.$$

$$[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}, \qquad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \qquad \text{with} \quad i, j \in \{x, y, z\},$$

and the formulas for commutators:

$$[a, b + c] = [a, b] + [a, c] \,,$$

$$[a, bc] = [a, b]c + b[a, c] \,.$$